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Conformal spectrum of the six-vertex model†

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Abstract. By way of a generalization of analytical methods recently introduced, the conformal spectrum of the six-vertex model with twisted boundary conditions is analysed. Nonlinear integral equations are derived, from which all scaling dimensions x and the spins s can be extracted analytically in terms of Rogers dilogarithms. For twist angle ϕ, the scaling dimensions of low-lying excitations are given by the so-called Gaussian dimensions

$$x = \frac{1-\gamma/\pi}{2}S^2 + \frac{1}{2(1-\gamma/\pi)}\left(m - \frac{\phi}{\pi}\right)^2$$

where γ is the crossing parameter, S the magnetization and m another integer characterizing the excited state. These results are applied to the calculation of critical exponents of the eight-vertex model. Further applications are pointed out.

1. Introduction

A major area of interest in statistical physics is the study of critical properties of models at phase transitions and, more specifically, the determination of the scaling dimensions x, from which the critical exponents can be deduced. Especially in two dimensions a wealth of important information can be gained from the assumption of conformal invariance of models at criticality [1, 2]. In this paper we study the critical two-dimensional six-vertex model and the associated spin-$\frac{1}{2}$ XXZ quantum chain.

The aim of many studies of critical systems is the identification of the underlying conformally invariant field theory by calculating the central charge c and the scaling dimensions x. For lattice models one may pursue two different methods. First, the energy levels of the corresponding quantum model on a finite chain will scale with the system size N like [3, 4]

$$E_0 = Ne_0 - \frac{\pi v}{6N}c + \sigma\left(\frac{1}{N}\right) \qquad E_x - E_0 = \frac{2\pi}{N}v\,x + \sigma\left(\frac{1}{N}\right) \tag{1.1}$$

where E_0 is the ground state energy, E_x are the energies of the low-lying excited states and v is the sound velocity of the elementary energy–momentum excitations. Equivalent formulae can be set up for the spectrum of the transfer matrix of the two-dimensional model [5]. Second, the quantum model can be studied on an infinite chain at finite temperature [6]. The low-temperature behaviour of the free energy density and the correlation lengths

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also allows for a calculation of c and all x [6–8]. This method, however, is more involved and will not be used in the present paper.

Of course, a lot is known about the critical six-vertex model either by analytical [9–11] or numerical methods [12, 13] based on (1.1). The dimensions x are given by the Gaussian spectrum. The analytic method for calculating finite-size corrections as developed in [9, 10] was applied to many models, cf [14] and references therein. However, it has a shortcoming as it fails for the higher spin-S XXZ chains [15, 16] and similar problems. In [17] an alternative procedure was developed which proved to be applicable to the $S = 1$ case thus opening a wide field of investigations [18–21]. In [17] the ground state energies of the quantum chains were treated. Here we want to show how excitations are dealt with and at the same time we shall give a comprehensive analytic study of the six-vertex model.

Another reason to revisit the six-vertex model is its importance for the investigation of the thermodynamics of the spin-$\frac{1}{2}$ XXZ chain. The standard approach to thermodynamics of integrable quantum chains [22, 23] is restricted to the calculation of the free energy, while correlation functions are out of reach. The situation is different in the quantum transfer matrix approach [24–26]. There the correlation lengths are accessible from the low-lying part of the spectrum of the quantum transfer matrix. Its evaluation is the main problem within this alternative approach. The necessary calculations are similar to those in section 3 below and lead to non-linear integral equations [27] which also are very similar to the equations for the finite-size corrections. For an analogous treatment of the thermodynamics of quantum chains related to RSOS models we refer to [8].

In section 2 we define the six-vertex model on a square lattice with a 'seam'. This is done to enable contact to be made with other models based on the Temperley–Lieb algebra [28, 29]. In section 3 we derive nonlinear integral equations from the Bethe ansatz equations. The integral equations turn out to allow for simple numerical solutions and for analytic studies of the large system asymptotics. The $1/N$ corrections are calculated analytically in section 4, from which the scaling dimensions are identified. The conclusion in section 5 contains a short overview on possible applications of the results derived in this paper.

2. The six-vertex model: definition and eigenvalue equation

We consider the six-vertex model on a square lattice. The arrow or spin variables on each bond of the lattice take the values $\pm\frac{1}{2}$. The statistical weights of the allowed vertices are given by the R-matrix $R_{\alpha}^{\beta}{}_{\gamma}^{\delta}(v)$ corresponding to a vertex with spins $\alpha, \beta, \gamma, \delta$ on the lower, upper, left and right bonds, respectively. The non-zero weights are

$$\begin{aligned} R_{1/2}^{1/2}{}_{1/2}^{1/2}(v) &= \sinh(\lambda/2 - v) \\ R_{-1/2}^{1/2}{}_{1/2}^{-1/2}(v) &= \sinh\lambda \end{aligned} \tag{2.1}$$

and those related to these by the symmetries

$$R_{\alpha}^{\beta}{}_{\gamma}^{\delta}(v) = R_{-\alpha}^{-\beta}{}_{-\gamma}^{-\delta}(v) = R_{\gamma}^{\delta}{}_{\alpha}^{\beta}(v) = R_{\beta}^{\alpha}{}_{\delta}^{\gamma}(v) \tag{2.2}$$

and the crossing symmetry

$$R_{\alpha}^{\beta}{}_{\gamma}^{\delta}(v) = R_{\gamma}^{\delta}{}_{-\beta}^{-\alpha}(-v) \tag{2.3}$$

where v denotes the spectral parameter. The 6-vertex model is critical for imaginary values of the crossing parameter λ: $\lambda = -\mathrm{i}\gamma$ with $\gamma \in [0, \pi[$. In most of the following we shall restrict ourselves to the 'repulsive range', $\gamma \in [0, \pi/2[$. This is done for the sake of a simple presentation of our analysis. The final results, however, have full validity. As in [12], we consider twisted boundary conditions by introducing a seam on the lattice, i.e. by changing the vertex weights in column N to $e^{2i\delta\phi} R_{\alpha}^{\beta}{}_{\gamma}^{\delta}(v)$. ϕ is called the twist angle.

As is well known, each eigenvalue $\Lambda(v)$ of the transfer matrix of the six-vertex model satisfies the functional equation [30]

$$\Lambda(v)q(v) = \omega^{-1}\Phi(v - i\gamma/2)\, q(v + i\gamma) + \omega\Phi(v + i\gamma/2)\, q(v - i\gamma) \tag{2.4}$$

with functions $\Phi(v)$ and $q(v)$ defined by

$$\Phi(v) = (\sinh v)^N \qquad q(v) = \prod_{j=1}^{n}(v - v_j) \tag{2.5}$$

and twist factor

$$\omega = e^{i\phi}. \tag{2.6}$$

The Bethe ansatz numbers (or moments) v_j, $j = 1, \ldots, n$, $0 \le n \le N/2$, have to satisfy the Bethe ansatz equations

$$p(v_j) = -1 \qquad j = 1, \ldots, n \tag{2.7}$$

where the function $p(v)$ is defined by

$$p(v) = \frac{1}{\omega^2}\frac{\Phi(v - i\gamma/2)\, q(v + i\gamma)}{\Phi(v + i\gamma/2)\, q(v - i\gamma)}. \tag{2.8}$$

One of the essential ingredients of our method is the identification of certain analyticity domains of the functions $\Phi(v)$, $q(v)$, $\Lambda(v)$, i.e. strips in the complex plane where these functions are *analytic* and *non-zero* (ANZ). The analytic properties of the eigenvalue functions were extensively studied in [31, 32], the results of which are used here. For details we refer directly to the original papers. For the ground state with $n = N/2$, real Bethe ansatz numbers analyticity domains are given by

$$\begin{array}{lllllll} \Phi(v) & \text{ANZ in} & 0 & < & \mathrm{Im}(v) & < & \pi \\ q(v) & \text{ANZ in} & -\pi & < & \mathrm{Im}(v) & < & 0 \\ \Lambda_0(v) & \text{ANZ in} & -\gamma/2 & \le & \mathrm{Im}(v) & \le & \gamma/2 \end{array} \tag{2.9}$$

where Λ_0 denotes the largest eigenvalue.

For the low-lying excitations there are $(N/2) - m$ real Bethe ansatz numbers with $m \in \mathbb{N}_+$. The remaining m moments either have a non-vanishing imaginary part, forming *strings* in the complex plane, or they are missing altogether.

In [31] it was shown that in the thermodynamic limit only two types of strings can occur:

- 2-strings, i.e. pairs of moments v_u,v_l with $|\mathrm{Im}(v_u)|$, $|\mathrm{Im}(v_l)| < \gamma$ and $\mathrm{Im}(v_u) - \mathrm{Im}(v_l) \simeq \gamma$, and
- 1-strings, i.e. single moments v_j with $\gamma < \mathrm{Im}(v_j) < \pi - \gamma$.

Each type of string gives rise to two real zeros Θ_1, Θ_2 of the corresponding eigenvalue function $\Lambda(v)$, henceforth called rapidities. This is also true for the 'missing' moments, which can be thought of as 1-strings with an infinite real part. As will be seen in the following sections, the occurrence of these rapidities leads to essential modifications of the calculations performed for the ground state in order to make them applicable also to the excitations. To summarize we note that excited states are characterized by patterns of strings and rapidities in the complex plane. In the following we shall denote the number of strings with positive (negative) real part by $k_\pm$ and the number of positive (negative) rapidities by $\nu_\pm$.

Finally we remark that the number of factors of the q function (2.5), i.e. the number of all real and complex moments in a given state, is obviously equal to $N/2 - S$, where S is the magnetization of the state. Furthermore every complex moment corresponds to a simple zero of $q(v)$ in the strip (2.9).

3. Nonlinear integral equations

The first main step of our analysis is to derive an integral equation for some auxiliary functions defined by

$$\begin{aligned} \mathfrak{a}(x) &:= 1/p(x - \mathrm{i}\gamma/2) & \mathfrak{A}(x) &:= 1 + \mathfrak{a}(x) \\ \overline{\mathfrak{a}}(x) &:= p(x + \mathrm{i}\gamma/2) & \overline{\mathfrak{A}}(x) &:= 1 + \overline{\mathfrak{a}}(x). \end{aligned} \tag{3.1}$$

Although the functions $\mathfrak{a}$ and $\overline{\mathfrak{a}}$ are related to one another by complex conjugation, it will prove to be convenient to treat them as independent of each other. First we observe from (2.5) and (2.8) that the functions (3.1) have simple asymptotics:

$$\begin{aligned} \mathfrak{a}(\pm\infty) &= \mathrm{e}^{2\mathrm{i}(\phi \pm S\gamma)} & \mathfrak{A}(\pm\infty) &= 1 + \mathfrak{a}(\pm\infty) \\ \overline{\mathfrak{a}}(\pm\infty) &= \mathrm{e}^{-2\mathrm{i}(\phi \pm S\gamma)} & \overline{\mathfrak{A}}(\pm\infty) &= 1 + \overline{\mathfrak{a}}(\pm\infty). \end{aligned} \tag{3.2}$$

To derive integral equations for our auxiliary functions, we define the Fourier transform $\mathcal{F}_k^{\mathcal{C}}\{f\}$ of a complex function $f(v)$ along the integration path $\mathcal{C}$ by

$$\mathcal{F}_k^{\mathcal{C}}\{f\} := \frac{1}{2\pi} \int_{\mathcal{C}} f(y) \mathrm{e}^{-\mathrm{i}ky}\, \mathrm{d}y \tag{3.3}$$

where the real part of the integration variable varies from $-\infty$ to ∞. If the path $\mathcal{C}$ is simply a straight line along the real axis, we just write $\mathcal{F}_k\{f\}$ instead of $\mathcal{F}_k^{\mathbb{R}}\{f\}$. Due to Cauchy's theorem one always has $\mathcal{F}_k^{\mathcal{C}}\{f\} = \mathcal{F}_k^{\widetilde{\mathcal{C}}}\{f\}$, as long as the paths $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ do not enclose singularities of $f(v)$, which will turn out to be extremely important for our further manipulations.

It is necessary to give a short qualitative account of the analyticity properties of the functions in the presence of rapidities and strings. To this end we observe that due to (2.4) and (2.8) the definition (3.1) can be written in the form

$$\mathfrak{a}(x) = \omega \frac{\Lambda(x - \mathrm{i}\gamma/2) q(x - \mathrm{i}\gamma/2)}{\Phi(x - \mathrm{i}\gamma) q(x + \mathrm{i}\gamma/2)} - 1. \tag{3.4}$$

From (3.4) it follows immediately that $\mathfrak{A}(\Theta + \mathrm{i}\gamma/2) = 0$ for each rapidity Θ, and similarly we have $\overline{\mathfrak{A}}(\Theta - \mathrm{i}\gamma/2) = 0$. Furthermore we conclude that for every complex moment v_j we have $\mathfrak{a}(v_j - \mathrm{i}\gamma/2) = \mathfrak{A}(v_j - \mathrm{i}\gamma/2) = \overline{\mathfrak{A}}(v_j + \mathrm{i}\gamma/2) = \infty$ as well as $\mathfrak{A}(v_j + \mathrm{i}\gamma/2) = \overline{\mathfrak{A}}(v_j - \mathrm{i}\gamma/2) = 0$. We remark that for finite systems the expression $\mathrm{Im}(v_u) - \mathrm{Im}(v_l) = \gamma$ for 2-strings is valid only up to certain corrections, so the zeros and poles of $\mathfrak{A}(v)$ and $\overline{\mathfrak{A}}(v)$ occurring halfway between the two moments forming the string are separated by a small distance.

This pattern of zeros and poles of the auxiliary functions forces us to perform certain Fourier transforms along deformed integration paths, in order to meet the requirements for the above-mentioned application of Cauchy's theorem. Therefore we introduce the path $\mathcal{L}$ which basically follows the real axis from $-\infty$ to ∞, but is deformed in such a way as to encircle the points $\Theta + \mathrm{i}\gamma/2$ and $v_j - \mathrm{i}\gamma/2$ clockwise. Here Θ denotes any of the real rapidities and v_j any of the 1-strings or upper moments of the 2-strings in the complex plane. Thus the path $\mathcal{C}$ encircles a zero of $\mathfrak{A}(x)$, whenever it passes a rapidity, and a zero of $\overline{\mathfrak{A}}(x)$ and a pole of $\mathfrak{a}(x)$ and $\mathfrak{A}(x)$, whenever it passes a string. In fact, whenever a 2-string occurs, one has to take care that the point $v_l + \mathrm{i}\gamma/2$ lies above the deformed path where v_l denotes again the lower of the two moments.

After these preliminary considerations, we can now tackle the actual derivation of the integral equation. First we write $\mathfrak{a}(x)$ and $\overline{\mathfrak{a}}(x)$ as

$$\begin{aligned}\mathfrak{a}(x) &= \omega^2(-)^{N/2-S}\frac{\Phi(x)}{\Phi(x+\mathrm{i}\pi-\mathrm{i}\gamma)}\frac{q(x-\mathrm{i}3\gamma/2)}{q(x+\mathrm{i}\gamma/2-\mathrm{i}\pi)}\\ \overline{\mathfrak{a}}(x) &= \omega^2(-)^{N/2-S}\frac{\Phi(x)}{\Phi(x+\mathrm{i}\gamma)}\frac{q(x+\mathrm{i}3\gamma/2-\mathrm{i}\pi)}{q(x-\mathrm{i}\gamma/2)}\end{aligned} \tag{3.5}$$

by using the πi-antiperiodicity to reduce the arguments of the functions $\Phi(v)$ and $q(v)$ to the strips (2.9). Then we apply the Fourier transform to the second logarithmic derivative of the first equation

$$\mathcal{F}_k[\ln\mathfrak{a}]'' = \left(1-\mathrm{e}^{(\gamma-\pi)k}\right)\mathcal{F}_k[\ln\Phi]'' + \left(\mathrm{e}^{\frac{3}{2}\gamma k}-\mathrm{e}^{(\pi-\frac{\gamma}{2})k}\right)\mathcal{F}_k^-[\ln q]'' \tag{3.6}$$

where the superscript '$-$' denotes an integration path along the real axis with a small negative imaginary part. The reader should convince himself that the paths $\mathcal{L}-\mathrm{i}3\gamma/2$ and $\mathcal{L}+\mathrm{i}(\gamma/2-\pi)$ do not enclose any singularities of $q(x)$, so that in (3.6) we are allowed to redeform them to straight lines and move them close to the real axis from below.

Now we want to exploit the fact that the only zeros of the eigenvalue function $\Lambda(v)$ in strip (2.9) are given by the rapidities Θ. We define the function $h(v)$ by

$$h(v) := \frac{1+p(v)}{q(v)} = \frac{1}{\omega}\frac{\Lambda(v)}{\Phi(v+\mathrm{i}\gamma/2)q(v-\mathrm{i}\gamma)} \tag{3.7}$$

which has zeros at $v=\Theta$ and poles at $v=v_j+\mathrm{i}\gamma$. We want to calculate the Fourier transform of $[\ln h]''$ in two different ways by using two different representations

$$\begin{aligned}h(x-\mathrm{i}\gamma/2) &= \frac{\mathfrak{A}(x)}{q(x-\mathrm{i}\gamma/2)\mathfrak{a}(x)}\\ h(x+\mathrm{i}\gamma/2) &= \frac{\overline{\mathfrak{A}}(x)}{q(x+\mathrm{i}\gamma/2-\mathrm{i}\pi)}.\end{aligned} \tag{3.8}$$

By the same reasoning as above one can derive the following expressions for $\mathcal{F}_k^+[\ln h]''$

$$\begin{aligned}\mathrm{e}^{\gamma/2k}\mathcal{F}_k^+[\ln h]'' &= -\mathrm{e}^{\gamma/2k}\mathcal{F}_k^-[\ln q]'' + \mathcal{F}_k^{\mathcal{L}}[\ln\mathfrak{A}]'' - \mathcal{F}_k^{\mathcal{L}}[\ln\mathfrak{a}]''\\ \mathrm{e}^{-\gamma/2k}\mathcal{F}_k^+[\ln h]'' &= \mathcal{F}_k^{\mathcal{L}}[\ln\overline{\mathfrak{A}}]'' - \mathrm{e}^{(\pi-\gamma/2)k}\mathcal{F}_k^-[\ln q]''\end{aligned} \tag{3.9}$$

where the superscripts '+' and '$-$' again denote integration paths along the real axis with small positive and negative imaginary parts, respectively. It is simple to solve these equations and (3.6) for $\mathcal{F}_k^{\mathcal{L}}[\ln\mathfrak{a}]''$ and $\mathcal{F}_k^-[\ln q]''$ in terms of $\mathcal{F}_k^{\mathcal{L}}[\ln\mathfrak{A}]''$ and $\mathcal{F}_k^{\mathcal{L}}[\ln\overline{\mathfrak{A}}]''$, obtaining

$$\mathcal{F}_k^{\mathcal{L}}[\ln\mathfrak{a}]'' = \frac{Nk}{1+\mathrm{e}^{-\gamma k}} + \frac{\sinh\left(\frac{1}{2}\pi-\gamma\right)k}{2\cosh\frac{1}{2}\gamma k\sinh\frac{1}{2}(\pi-\gamma)k}\left[\mathcal{F}_k^{\mathcal{L}}[\ln\mathfrak{A}]'' - \mathrm{e}^{(\gamma-\epsilon)}\mathcal{F}_k^{\mathcal{L}}[\ln\overline{\mathfrak{A}}]''\right] \tag{3.10}$$

$$\mathcal{F}_k^-[\ln q]'' = \frac{Nk\,\mathrm{e}^{-\frac{\pi}{2}k}}{4\sinh\frac{\pi}{2}k\cosh\frac{\gamma}{2}k} + \frac{\mathrm{e}^{-\frac{\pi+\gamma}{2}k}}{4\cosh\frac{\gamma}{2}k\sinh\frac{\pi-\gamma}{2}k}\left[\mathrm{e}^{\gamma k}\mathcal{F}_k^{\mathcal{L}}[\ln\overline{\mathfrak{A}}]'' - \mathcal{F}_k^{\mathcal{L}}[\ln\mathfrak{A}]''\right]. \tag{3.11}$$

To proceed we take the inverse Fourier transform of (3.10) to obtain†

$$[\ln \mathfrak{a}]''(x) = N\left[\ln\tanh\left(\frac{\pi x}{2\gamma}\right)\right]'' + \int_{\mathcal{L}}\left\{F(x-y)\,[\ln\mathfrak{A}]''(y) - F(x-y+\mathrm{i}\epsilon-\mathrm{i}\gamma)\left[\ln\overline{\mathfrak{A}}\right]''(y)\right\}\mathrm{d}y \qquad (3.12)$$

where $F(x)$ is defined by

$$F(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sinh\left(\frac{\pi}{2}-\gamma\right)k}{2\cosh\frac{\gamma}{2}k\,\sinh\frac{\pi-\gamma}{2}k}\mathrm{e}^{ikx}\mathrm{d}k. \qquad (3.13)$$

Now we write down the final integral equation by integrating (3.12) twice obtaining

$$\ln\mathfrak{a}(x) = N\ln\left[\tanh\frac{\pi x}{2\gamma}\right] + \int_{\mathcal{L}}\left[F(x-y)\ln\mathfrak{A}(y) - F(x-y+\mathrm{i}\epsilon-\mathrm{i}\gamma)\ln\overline{\mathfrak{A}}(y)\right]\mathrm{d}y + C \qquad (3.14)$$

where the integration constant C can be determined from the asymptotic behaviour (3.2), see below. Equation (3.14) is exact for all finite system sizes N.

4. Analytic calculation of $1/N$ corrections

To determine the finite-size corrections to the largest eigenvalue in terms of the functions $\mathfrak{A}(x)$ and $\overline{\mathfrak{A}}(x)$, (2.4) is written in the form

$$\Lambda(x-\mathrm{i}\gamma/2) = (-)^{N/2-S}\frac{1}{\omega}\Phi(x-\mathrm{i}\gamma)\frac{q(x+\mathrm{i}\gamma/2-\mathrm{i}\pi)}{q(x-\mathrm{i}\gamma/2)}\mathfrak{A}(x). \qquad (4.1)$$

Fourier transforming, inserting (3.11), and then manipulating the resulting equation in much the same way as before, one eventually finds

$$\ln\Lambda(x-\mathrm{i}\gamma/2) = \ln\Phi(x-\mathrm{i}\gamma) - N\mathrm{i}\int_{-\infty}^{\infty}\frac{\sinh\left(\frac{\pi-\gamma}{2}\right)k\sin xk}{2k\sinh\frac{\pi}{2}k\cosh\frac{\gamma}{2}k}\mathrm{d}k + \frac{\mathrm{i}}{2\gamma}\int_{\mathcal{L}}\frac{\ln\mathfrak{A}(y)+\ln\overline{\mathfrak{A}}(y)}{\sinh\frac{\pi}{\gamma}(x-y+\mathrm{i}\epsilon)}\mathrm{d}y + \left(S-\frac{\Delta\hat{\nu}}{2}-\frac{N}{2}\right)\pi\mathrm{i} \qquad (4.2)$$

with abbreviations

$$\hat{\nu}_\pm := \nu_\pm - 2k_\pm \qquad \Delta\hat{\nu} := \hat{\nu}_+ - \hat{\nu}_-. \qquad (4.3)$$

The bulk contribution is entirely contained in the first line of (4.2) and the finite-size corrections are given in terms of the function $\mathfrak{A}(x)$ alone. The lattice momentum P_0 of the state can be read off as

$$P_0 = \left(S-\frac{\Delta\hat{\nu}}{2}-\frac{N}{2}\right)\pi. \qquad (4.4)$$

† Actually, whenever strings occur in the complex plane, equation (3.12) cannot be established by simply transforming back (3.10) because the k-integral need not necessarily be convergent any longer. In that case one can obtain (3.12) by first expressing the Fourier transforms along $\mathcal{L}$ by 'usual' Fourier transforms (using the residue theorem), then transforming back the resulting equation and finally eliminating the additive terms by deforming the integration paths again.

Again equation (4.2) is exact for all finite system sizes N.

To make further progress we now perform a suitable scaling limit by substituting the arguments x of the functions $\mathfrak{a}$ and $\mathfrak{A}$ by $\pm(x + \ln N)$ and letting $N \to \infty$; the scaled functions are denoted by $a_\pm$ and $A_\pm$ respectively. In that way we obtain the scaled equation

$$\ln a_\pm(x) = -2\mathrm{e}^{-x} + F_1 *_\mathcal{L} \ln A_\pm - F_2 *_\mathcal{L} \ln \overline{A}_\pm + C_\pm \tag{4.5}$$

where F_1 and F_2 are defined by

$$\begin{aligned} F_1(y) &:= \frac{\gamma}{\pi} F\left(\frac{\gamma}{\pi} y\right) \\ F_2(y) &:= \frac{\gamma}{\pi} F\left(\frac{\gamma}{\pi} y \pm \mathrm{i}(\epsilon - \gamma)\right) \end{aligned} \tag{4.6}$$

and $f *_\mathcal{L} g$ denotes the convolution of two functions f and g along $\mathcal{L}$

$$(f *_\mathcal{L} g)(x) := \int_\mathcal{L} f(x-y) g(y) \mathrm{d}y. \tag{4.7}$$

The values of $C_\pm$ are determined from the asymptotics of the auxiliary functions

$$\begin{aligned} \ln a_\pm(\infty) &= 2\mathrm{i}(\phi \pm S\gamma) \\ \ln \overline{a}_\pm(\infty) &= -2\mathrm{i}(\phi \pm S\gamma) \\ \ln A_\pm(\infty) &= \ln\left(1 + \mathrm{e}^{2\mathrm{i}(\phi \pm S\gamma)}\right) \pm (k_\pm - \nu_\pm) 2\pi\mathrm{i} \\ \ln \overline{A}_\pm(\infty) &= \ln\left(1 + \mathrm{e}^{-2\mathrm{i}(\phi \pm S\gamma)}\right) \mp k_\pm 2\pi\mathrm{i} \end{aligned} \tag{4.8}$$

yielding

$$C_\pm = \frac{\pi\mathrm{i}}{\pi - \gamma}\left[\phi \pm \left(S - 2\hat{\nu}_\pm\right)\gamma \pm \hat{\nu}_\pm \pi\right]. \tag{4.9}$$

The same scaling limit can be performed for the second integral on the RHS of (4.2), leading to†

$$\begin{aligned} \frac{\mathrm{i}}{2\gamma}\int_\mathcal{L} \frac{\ln A(y) + \ln \overline{A}(y)}{\sinh \frac{\pi}{\gamma}(x - y + \mathrm{i}\epsilon)} \mathrm{d}y &= -\frac{\mathrm{i}}{\pi N} \mathrm{e}^{\frac{\pi}{\gamma}x} \int_\mathcal{L} \left(\ln A_+(y) + \ln \overline{A}_+(y)\right) \mathrm{e}^{-y} \mathrm{d}y \\ &\quad + \frac{\mathrm{i}}{\pi N} \mathrm{e}^{-\frac{\pi}{\gamma}x} \int_\mathcal{L} \left(\ln A_-(y) + \ln \overline{A}_-(y)\right) \mathrm{e}^{-y} \mathrm{d}y + \sigma\left(\frac{1}{N}\right) \end{aligned} \tag{4.10}$$

from which the scaling dimensions and spins [5] can be read off as

$$\begin{aligned} x &= -\frac{1}{2\pi^2}\left[\int_\mathcal{L} \left(\ln A_+(y) + \ln \overline{A}_+(y)\right) \mathrm{e}^{-y} \mathrm{d}y + \int_\mathcal{L} \left(\ln A_-(y) + \ln \overline{A}_-(y)\right) \mathrm{e}^{-y} \mathrm{d}y\right] + \frac{1}{12} \\ s &= -\frac{1}{2\pi^2}\left[\int_\mathcal{L} \left(\ln A_+(y) + \ln \overline{A}_+(y)\right) \mathrm{e}^{-y} \mathrm{d}y - \int_\mathcal{L} \left(\ln A_-(y) + \ln \overline{A}_-(y)\right) \mathrm{e}^{-y} \mathrm{d}y\right] \end{aligned} \tag{4.11}$$

where we have used that the central charge of the six-vertex model (without twist) is $c = 1$ and the amplitude of the finite-size correction for the groundstate energy is given by [17]

$$c(\phi) = 1 - \frac{6(\phi/\pi)^2}{1 - \gamma/\pi}. \tag{4.12}$$

† Here the reason for the particular choice of the phase factors in (4.8) becomes evident. To guarantee the convergence of the integrals on the RHS of (4.10) one has to take care that the expressions $\ln A_\pm(y) + \ln \overline{A}_\pm(y)$ exponentially tend to zero for $y \to -\infty$, which is achieved by (4.8).

Fortunately, the integrals in (4.11) can be determined without explicitly solving the integral equation (4.5) for $a_\pm$. After some algebraic manipulations of (4.5) and of the corresponding equation for $\overline{a}_\pm$, and using certain symmetries of the functions F_1, F_2 [17] one finds

$$4\left\{\int_{\mathcal{L}} \mathrm{e}^{-x}\left(\ln A_\pm(x)+\ln \overline{A}_\pm(x)\right)\mathrm{d}x\right\}$$
$$= 2L_+\left(a_\pm(\mathcal{L})\right)+2L_+\left(\overline{a}_\pm(\mathcal{L})\right)-C_\pm\left(\ln \overline{A}_\pm(\infty)-\ln A_\pm(\infty)\right) \qquad (4.13)$$

where the dilogarithmic function $L_+(C)$ is defined by

$$L_+(C) := \frac{1}{2}\int_C\left[\frac{\ln(1+y)}{y}-\frac{\ln y}{1+y}\right]\mathrm{d}y \qquad (4.14)$$

and C is a path starting at 0. For a straight integration path C one can write $L_+(z)$, where z is the endpoint of C. Note that the value of $L_+(C)$ in general depends on the homotopy class of C and not only on its endpoint. L_+ is related to the standard Rogers dilogarithm L by $L_+(z) = L(z/1+z)$. Using (4.8), (4.9), (4.13), and the identities

$$2L_+\left(a_\pm(\mathcal{L})\right)+2L_+\left(\overline{a}_\pm(\mathcal{L})\right)=\frac{\pi^2}{3}+4\pi\hat{\nu}_\pm\left(\pm\phi+S\gamma-\hat{\nu}_\pm\right)-8\pi^2k_\pm\left(\nu_\pm-k_\pm\right) \qquad (4.15)$$

(following from formula (A.11) of the appendix) and finally inserting everything into (4.11) we find

$$x=\frac{1-\gamma/\pi}{2}S^2+\frac{1}{2(1-\gamma/\pi)}\left[\left(m-\frac{\phi}{\pi}\right)^2\right]+k+\overline{k}$$
$$s=S\left(m-\frac{\phi}{\pi}\right)+k-\overline{k} \qquad (4.16)$$

with $m=\Delta\hat{\nu}/2\in\mathbb{Z}$ and integers k, $\overline{k}$ defined by

$$k := k_+(\nu_+-k_+)\in\mathbb{N}$$
$$\overline{k} := k_-(\nu_--k_-)\in\mathbb{N}. \qquad (4.17)$$

Up to now we have tacitly assumed that all rapidities and strings are positioned outside the distribution of real Bethe ansatz numbers. If this is not the case, formula (4.15) becomes much more complicated. As mentioned above, the values of $L\left(a_\pm(\mathcal{L})\right)$ and $L\left(\overline{a}_\pm(\mathcal{L})\right)$ depend on the homotopy classes of $a_\pm(\mathcal{L})$ and $\overline{a}_\pm(\mathcal{L})$, respectively. More precisely, they depend on the way these paths encircle the singularities of the integrand in (4.14) at 0 and -1, thus singling out different branches of the dilogarithms. In fact the path $a_+(\mathcal{L})$ ($a_-(\mathcal{L})$) encircles the origin clockwise (counter-clockwise) whenever $\mathcal{L}$ passes a real Bethe-ansatz number, and it encircles the point -1 clockwise (counter-clockwise) whenever $\mathcal{L}$ encircles a zero of $A_\pm(x)$ and counter-clockwise (clockwise) whenever a pole of $A_\pm(x)$ is encircled. Analogous statements are true for $\overline{a}_\pm(\mathcal{L})$.

Using these general properties we have carried through the foregoing calculation also for the more general case when the rapidities and strings are allowed to be interspersed with real Bethe ansatz numbers; it turns out that one does not find any new scaling dimensions but the values of the numbers k, $\overline{k}$ in (4.16) are modified. One can show that they are still non-negative, the corresponding states belonging to the same conformal tower as before. We confine ourselves to give the formulae for k, $\overline{k}$ for the special case when only rapidities but no complex moments are present. In that particular case one obtains (using (A.13))

$$k = \sum_{i=1}^{\nu_+} \sum_{j=i}^{\nu_+} k_i^+ \in \mathbb{N}$$
$$\bar{k} = \sum_{i=1}^{\nu_-} \sum_{j=i}^{\nu_-} k_i^- \in \mathbb{N} \tag{4.18}$$

where the non-negative integers $k_i^\pm$ denote the distances of the ith positive or negative rapidity from the right or left edge of the distribution of real Bethe ansatz numbers, respectively.

Some of the scaling dimensions and spins given by (4.16) corresponding to low-lying excitations have been calculated numerically in [12] and analytically in [33, 34]. Furthermore, equation (4.16) confirms the conjecture concerning the full operator content of the XXZ chain with twisted boundary conditions obtained numerically in [13]. For the particular case of $\phi = 0$, the result (4.16) has already been analytically obtained in [10] and [11]. In [11] one can also find analytical results for the scaling dimensions of the Potts model corresponding to the general equation (4.16).

5. Conclusion

We have presented an analysis of the $1/N$ corrections to the eigenvalues of the six-vertex model transfer matrix. From these data the central charge and the scaling dimensions of the model could be derived.

In particular we have rewritten the Bethe ansatz equations in terms of non-linear integral equations which are exact for any finite system size N and which admit simple numerical calculations and analytic studies of the large N asymptotics. The calculations have been given for anisotropy $\gamma < \pi/2$ which allowed for a simple presentation. The results, however, are valid throughout the entire critical regime $0 < \gamma < \pi$.

A first application of these results is the calculation of the critical exponents of some observables in the eight-vertex model. In general, the exponents ν and η are related to the leading thermal and electric exponents x_ϵ and x_e by $\nu = 1/(2 - x_\epsilon)$ and $\eta_e = 2x_e$, respectively. With $x_\epsilon = x(S = 2, m = 0, \Phi = 0)$ and $x_e = x(S = 1, m = 0, \Phi = 0)$ in (4.16) we obtain

$$\nu = \frac{\pi}{2\gamma} \qquad \eta_e = \frac{\pi - \gamma}{\pi} \tag{5.1}$$

from which α, β_e, γ_e and δ_e can be derived by employing scaling relations. These coincide with Baxter's results [30]. The continuous variation (with γ) can be understood from the existence of a marginal operator with $x = 2$ corresponding to the four-spin interaction. The subscript 'e' ('electric') refers to critical exponents of the model found in the vertex formulation.

The 'equivalent' IRF ('interaction-round-a-face') version [30] contains more observables, one of them being the magnetization. In order to calculate the additional critical exponents we are led to study the finite-size spectrum of the IRF transfer matrix. Equivalently, we consider the partition function Z_I of the eight-vertex model in Ising variables on a finite square lattice of size $N \times M$ (N and M even). Using the standard mapping [30] but keeping track of the correct boundary conditions, we find that Z_I equals twice the partition function in vertex formulation under the restriction that the total number of 'up' arrows in each row and the total number of 'right' arrows in each column are even. We can implement one of the restrictions by introducing a suitable seam in one of the columns. Eventually we find

$$Z_{\rm I} = 2Z_{\rm V}^{\rm even,even} = Z_{\rm V}^{\rm even}(\phi = 0) + Z_{\rm V}^{\rm even}\left(\phi = \frac{\pi}{2}\right) \tag{5.2}$$

where the last superscripts remind us that the total number of 'up' arrows in each row is even. From this formula we directly infer that the eigenvalues of the IRF transfer matrix are given by those of the 'vertex' transfer matrix with either $\phi = 0$ or $\phi = \pi/2$, in each case under the condition of an even number of up arrows in the corresponding eigenstates. (A similar relation exists for the IRF transfer matrix with antiperiodic boundary conditions and the vertex transfer matrices with $\phi = 0$ and $\phi = \pi/2$ under the condition of an odd number S.) This completely determines the spectrum of the IRF transfer matrix. Unfortunately, for no integer parameters S and m, the known magnetic scaling dimension $x_m = 1/8$ is given by (4.16) as the result is always a function of γ. It turns out that the reason for this lies in the particular gauge (2.1) we have used. The corresponding IRF model does not decouple into two Ising models at the special value of the crossing parameter $\gamma = \pi/2$. This can be achieved, however, by using (2.1) with the gauge transformation $\exp\left(\mathrm{i}\frac{\pi}{2}S^x\right)$ on each bond such that (5.2) still holds. For the calculation of the scaling dimensions it is possible to stay in the gauge (2.1) yet involving a vertical seam with operators $\exp(\mathrm{i}\pi S^y)$ instead of $\exp(\mathrm{i}\pi S^z)$. This situation is not covered by our approach, but was treated in [13, 12] where the magnetic exponent $x_m = 1/8$ actually appears in accordance with [35].

Some other models such as the Ashkin–Teller model are related to the six-vertex model by a Temperley–Lieb equivalence [28, 29, 30]. This implies that sectors of the quantum Ashkin–Teller model can be mapped onto certain sectors of the XXZ chain with possibly non-zero twist [12]. The thermal and electric exponents are then given by $x_\epsilon = x(S = 0, m = 1, \Phi = 0)$ and $x_e = x(S = 0, m = 0, \Phi = \pi/2)$ such that

$$\alpha = \frac{2-2y}{3-2y} \qquad \beta_e = \frac{1}{12-8y} \qquad y = \frac{2\gamma}{\pi}. \tag{5.3}$$

For details and further applications to the Potts model see, for instance, [11, 12].

Another application of our results is the treatment of the thermodynamics of the spin-$\frac{1}{2}$ XXZ chain within the quantum transfer matrix approach [24, 8]. The largest eigenvalue of this matrix and the next-leading ones yield the free energy and the correlation lengths, respectively. The study of these problems involves techniques similar to those employed in the main body of this paper, however, applied to the transfer matrix of a six-vertex model with inhomogeneity. Details of this investigation will be published elsewhere.

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Appendix

In this appendix we derive some formulae for dilogarithmic integrals.

The real dilogarithmic function L_+ is given by

$$L_+(x) = \frac{1}{2}\int_0^x \left[\frac{\ln(1+y)}{y} - \frac{\ln y}{1+y}\right] \mathrm{d}y = L\left(\frac{x}{1+x}\right) \tag{A.1}$$

where L is the Rogers dilogarithm. It satisfies the functional equation [36]

$$L_+(x) + L_+\left(\frac{1}{x}\right) = \frac{\pi^2}{6}. \tag{A.2}$$

Definition (A.1) can be extended to arbitrary integration paths $\mathcal{C}$ with starting point 0 and final point $x \in \mathbb{C}$

$$L_+(\mathcal{C}) = \frac{1}{2}\int_{\mathcal{C}}\left[\frac{\ln(1+y)}{y} - \frac{\ln y}{1+y}\right]\mathrm{d}y. \tag{A.3}$$

The notation $L_+(\mathcal{C})$ indicates that in general the value of the dilogarithmic integral depends on the homotopy class of $\mathcal{C}$ and not only on its endpoint as was the case in (A.1). However, as long as $\mathcal{C}$ does not encircle the point -1, we still have $L_+(\mathcal{C}) = L_+(x)$.†

Let us now consider a path $\mathcal{C}$ which starts at 0 and then encircles the point -1 (and 0) n times in a clockwise sense before finally arriving at x. (Such a path corresponds to $a_+(\mathcal{L})$, if n rapidities are present.) For $j = 1, \ldots, n$ we denote by $\mathcal{C}^j$ the path which also starts at 0, but first encircles the origin j times clockwise, before encircling -1 clockwise $n-j$ times and ending at x. By $\mathcal{C}_j$ we denote a path starting at -1, encircling the origin j times clockwise, and ending at x. According to these properties we have

$$L_+(\mathcal{C}^n) = L_+(x) \tag{A.4}$$

and moreover

$$\int_{\mathcal{C}_j}\frac{1}{y}\mathrm{d}y = \ln(x) - (2j+1)\pi\mathrm{i} \tag{A.5}$$

where $\ln(x)$ denotes the branch of the logarithm which is real for positive x. The higher branches are denoted by $\ln^i(x)$ $i = 1, 2, \ldots$

Now we want to establish a functional equation for $L(\mathcal{C})$ analogous to the one given in (A.2). Taking care of the branch cut of $\ln(1+y)$ along $]-\infty, -1[$ and of the simple pole of $\ln y/(1+y)$ at -1 ,we obtain

$$\begin{aligned}
2L_+(\mathcal{C}) &= \int_{\mathcal{C}}\left[\frac{\ln(1+y)}{y} - \frac{\ln y}{1+y}\right]\mathrm{d}y \\
&= \int_{\mathcal{C}^1}\frac{\ln(1+y)}{y}\mathrm{d}y - 2\pi\mathrm{i}\int_{\mathcal{C}_{n-1}}\frac{1}{y}\mathrm{d}y - \int_{\mathcal{C}^1}\frac{\ln y}{1+y}\mathrm{d}y + 2\pi\mathrm{i}\ln^{n-1}(-1) \\
&= \int_{\mathcal{C}^2}\frac{\ln(1+y)}{y}\mathrm{d}y - 2\pi\mathrm{i}\left[\int_{\mathcal{C}_{n-2}}\frac{1}{y}\mathrm{d}y + \int_{\mathcal{C}_{n-1}}\frac{1}{y}\mathrm{d}y\right] \\
&\quad - \int_{\mathcal{C}^2}\frac{\ln y}{1+y}\mathrm{d}y + 2\pi\mathrm{i}\left[\ln^{n-2}(-1) + \ln^{n-1}(-1)\right] \\
&= \int_{\mathcal{C}^n}\frac{\ln(1+y)}{y}\mathrm{d}y - 2\pi\mathrm{i}\sum_{j=1}^{n}\int_{\mathcal{C}_{n-j}}\frac{1}{y}\mathrm{d}y - \int_{\mathcal{C}^n}\frac{\ln y}{1+y}\mathrm{d}y + 2\pi\mathrm{i}\sum_{j=1}^{n}\ln^{n-j}(-1).
\end{aligned} \tag{A.6}$$

Using (A.4) and (A.5) and

$$\ln^j(-1) = (2j+1)\pi i, \tag{A.7}$$

† Due to $\mathrm{Res}_{y=0}\frac{\ln(1+y)}{y} = 0$ and $\lim_{\epsilon\to 0}\int_{|y|=\epsilon}\frac{\ln y}{1+y}\mathrm{d}y = 0$, $\mathcal{C}$ may encircle the origin arbitrarily many times.

(A.6) can be written as

$$2L_+(\mathcal{C}) = 2L_+(x) - 2\pi i\left[\sum_{j=1}^{n}\ln(x) - (2(n-j)+1)\pi i\right]$$
$$+2\pi i\left[\sum_{j=1}^{n}(2(n-j)+1)\pi i\right]$$
$$= 2L_+(x) - 2n\pi i\ln(x) - 4n^2\pi^2. \tag{A.8}$$

Thus we have with (A.2)

$$2L_+(\mathcal{C}) + 2L_+\left(\frac{1}{x}\right) = \frac{\pi^2}{3} - 2n\pi i\ln(x) - 4n^2\pi^2. \tag{A.9}$$

Let us now consider the case when $\mathcal{C}$ encircles the point -1 not only n times clockwise but also k times counter-clockwise (corresponding to $a_+(\mathcal{L})$ with n rapidities and k strings present). A formula for that case can immediately be obtained by substituting $n \to n-k$ thus yielding

$$2L_+(\mathcal{C}) = 2L_+(x) - 2(n-k)\pi i\ln(x) - 4(n-k)^2\pi^2. \tag{A.10}$$

Suppose now that we calculate a second dilogarithmic integral along a path $\tilde{\mathcal{C}}$ from 0 to $\frac{1}{x}$ which encircles the point -1 k times counter-clockwise (corresponding to $\overline{a}_+(\mathcal{L})$). Using (A.8) (with $n \to k$), (A.10), and (A.2) we obtain

$$2L(\mathcal{C}) + 2L\left(\tilde{\mathcal{C}}\right) = \frac{\pi^2}{3} - 2(n-2k)\pi i\ln(x) - 4(n^2 - 2nk + 2k^2)\pi^2. \tag{A.11}$$

Finally we wish to consider the situation when the path $\mathcal{C}$ is allowed to encircle the origin (but not -1) k_j times clockwise between the jth and the $j+1$th clockwise encircling of -1 (there must not be any counter-clockwise encirclings of -1 now) corresponding to the occurrence of real Bethe ansatz numbers between the rapidities. Instead of (A.6) we have then

$$2L_+(\mathcal{C}) = \int_{\mathcal{C}^n}\frac{\ln(1+y)}{y}\mathrm{d}y - 2\pi i\sum_{j=1}^{n}\int_{\mathcal{C}_{s(j)}}\frac{1}{y}\mathrm{d}y - \int_{\mathcal{C}^n}\frac{\ln y}{1+y}\mathrm{d}y + 2\pi i\sum_{j=1}^{n}\ln^{s(j)}(-1) \tag{A.12}$$

with $s(j) = \sum_{l=j}^{n} k_l$. This gives, taking account of (A.4), (A.5), and (A.7),

$$2L_+(\mathcal{C}) = 2L_+(x) - 2n\pi i\ln(x) - 4\left[n^2 + 2\sum_{j=1}^{n}\sum_{l=j}^{n}k_l\right]\pi^2. \tag{A.13}$$

To conclude, we mention that formulae analogous to (A.8), (A.10) and (A.13) but valid for paths $\mathcal{C}$ which encircle the points 0 and -1 in the opposite sense than assumed above (corresponding to $a_-(\mathcal{L})$ and $\overline{a}_-(\mathcal{L})$), can be obtained by simply inverting the sign of the second term on the RHS of these equations from $-$ to $+$.

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